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Multiple Fourier Method for Plate Bending

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FOR the differential equation

$$\nabla^4 w = q/D \tag{1}$$

describing the bending of thin plates, a series solution may be found of the form,[1]

$$w = w_p + \sum_{j=1,2,3}^{\infty} a_j w_j \tag{2}$$

q is the transverse load, D the plate stiffness, w_p the particular solution of Eq. (1), w_j the homogeneous solutions forming a complete set, and a_j the constant coefficients determined by the boundary conditions,

$$G_1(w) = 0 = G_2(w) \tag{3}$$

The differential operators $G_{1,2}$ produce deflections, slopes, moments and shears depending on the type of boundary considered.

If the plate region is simply connected and star shaped, then any function defined along the boundary may be described uniquely by the angle θ from the datum of some appropriately selected origin. Hence the function at the boundary may be represented as a Fourier series in θ. Thus the terms $G_{1,2}(w_p)$ may be represented by Fourier series as also may each of the series terms $G_{1,2}(w_j)$. Equations (2) and (3) may then be transformed to

$$\sum_{j=1,2,3}^{\infty} a_j \left[\sum_{n=1,2,3}^{\infty} \epsilon_n(\alpha_{1,2jn} \cos(n-1)\theta + \beta_{1,2jn} \sin(n-1)\theta) \right] = - \sum_{n=1,2,3}^{\infty} \epsilon_n(\alpha_{1,2pn} \cos(n-1)\theta + \beta_{1,2pn} \sin(n-1)\theta) \tag{4a}$$

Table 1 Values of α for $w_{\max} = \alpha q a^4/D$ for a uniformly, loaded simply supported rectangular plate

$b/a = 1$	0.0040636 (6)	0.0040632 (8)	0.0040624 (10)	0.0040624 (∞)
$b/a = 3$	0.012276 (8)	0.012236 (12)	0.012233 (16)	0.012233 (∞)
$b/a = 6$	0.013010 (24)			0.013010 (∞)

Received January 12, 1971.

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Table 2 Values of α for $w = \alpha q a^4/256D$ across the diagonal between corners of a uniformly loaded simply supported L-shaped plate

16d	0	1	3	5	7	9	11	13	15	16
Finite difference[4]	0	0.1	0.9	2.1	3.3	4.3	4.5	3.8	1.7	0
Integral equation[5]	...	0.1	0.9	2.1	3.4	4.5	4.8	4.2	2.5	...
Multiple Fourier	0	0.1	0.9	2.1	3.4	4.5	4.9	4.5	3.0	1.7

where the Fourier coefficients are given by

$$\begin{aligned} \alpha_{1,2jn} &= \frac{2}{s}\int_0^s G_{1,2}(w_j) \cos\frac{2\pi}{s}(n-1)\theta d\theta \\ \beta_{1,2jn} &= \frac{2}{s}\int_0^s G_{1,2}(w_j) \sin\frac{2\pi}{s}(n-1)\theta d\theta \\ \alpha_{1,2pn} &= \frac{2}{s}\int_0^s G_{1,2}(w_p) \cos\frac{2\pi}{s}(n-1)\theta d\theta \\ \beta_{1,2pn} &= \frac{2}{s}\int_0^s G_{1,2}(w_p) \sin\frac{2\pi}{s}(n-1)\theta d\theta \end{aligned} \tag{4b}$$

ϵ_n is a constant taking the value $\frac{1}{2}$ for $n = 1$, otherwise the value 1, and s is the periodicity $\leq 2\pi$. Equating coefficients in $\cos n\theta$ and $\sin n\theta$ in Eqs. (4a) the coefficients a_j may be determined from the matrix of linear equations

$$\begin{bmatrix} \epsilon_n \alpha_{1,2jn} \\ \epsilon_n \beta_{1,2jn} \end{bmatrix} \cdot [a_j] = - \begin{bmatrix} \epsilon_n \alpha_{1,2pn} \\ \epsilon_n \beta_{1,2pn} \end{bmatrix} \tag{5}$$

In practical numerical requirements of the solution of Eq. (5) it is necessary to truncate the series n at some value $2N$ and the series j at $4N$. The accuracy of solution will depend on the degree of convergence of the Fourier series for $2N$ terms. Smoothness, differentiability, discontinuities and periodicity of the functions are the well known governing factors of convergence. The rates of convergence of the Fourier series are related to the rate of convergence of the series in Eq. (2). A poor selection of series solutions in Eq. (2) and position of an origin(s) may produce very slow convergence. But, in any event, an increase in the number of terms will provide better solutions, except perhaps at points of discontinuity in boundary curvature ρ and transitions in boundary conditions and boundary loading.

For simplicity the Eqs. (4a, 4b, and 5) defining the Multiple Fourier method[2] were presented for a simply connected region. However, they are also valid for multiply connected regions, the α and β becoming summations of the form (4b) over the number of boundaries, and the number of terms in the Fourier series associated with one boundary possibly differing from those for another. Even though the method is simple the author has not been able to find published work for plate bending problems, although Pickett[3] intimates the use of a procedure essentially the same as the method described here but gives details only in the case of a classical single Fourier analysis for mixed polar and rectangular coordinate series.

Taking the $\Sigma^{2N} a_j w_j$ to be the polar series

$$\sum_{m=0,1,2}^{N-1} (A_m + B_m r^2) r^m \cos m\theta$$

for symmetrical uniformly loaded plates with an origin for radius r and angle θ in the central portion of the plate, the Multiple Fourier Method reduces to the classical single Fourier analysis for ring shaped plates[1] when boundary conditions are continuous.

The $2N(2N + 1)$ integrations of Eqs. (4a) and (4b) have been performed numerically using Simpson's rule in the examples which follow, considerable savings in calculation being achieved by tabulating repetitive terms. In the case of continuous boundary conditions further saving is possible by

Table 3 Values of α for $w_{\max} = qa^4/64D$, $\nu = 0.3$ for a uniformly loaded clamped and simply supported circle

Least squares collocation[6]	Multiple Fourier method				Boundary moments[7]
1.504 (20)	1.499 (42)	1.491 (62)	1.486 (82)	1.481 (122)	1.475 (20)

determining Fourier expansions only for boundary values of r, $1/r$, $\log r$ and finite $1/\rho$ if required, and determining the requisite expansions of Eqs. (4a) and (4b) by raising r and $1/r$ to the appropriate powers and converting the resultant trigonometrical polynomials by addition theorems.

For a uniformly loaded simply supported rectangular plate of dimensions $b \times a$, maximum deflection coefficients are shown in Table 1. The figures in brackets denote the number of nonvanishing terms used in the aforementioned polar series.

A more difficult problem of a uniformly loaded simply supported symmetrical L-shaped plate[4,5] produced the results of Table 2 for deflection across the diagonal between corner points, starting from the outside corner at point 0 and finishing at the re-entrant corner at point 16. Here, a, is the width of the plate with maximum length of $3a$, and d is the proportion of diagonal length traversed. The series above has been truncated at $m = 10$. The results for deflection (and also for moment) are in good agreement with the coupled integral method except near the re-entrant corner. Doubling the number of terms has very little effect on the results, suggesting that within practical limits the correct zero boundary deflection could not be obtained with the series solution adopted. This same problem tackled by a boundary collocation method at this maximum value of $m = 20$ yields very unstable results and far removed from the correct values.

A further problem of a uniformly loaded circular plate clamped and simply supported on alternating quadrants illustrates poor conditioning due to the discontinuity of boundary conditions. Since only $w = 0$ is a common boundary condition a single Fourier analysis cannot be used. Leissa and Clausen[6] solve the problem using least squares point matching and Conway and Farnham[7] use collocation for a boundary distribution of concentrated moments over the clamped segments. Central deflection coefficients for Poisson's ratio, $\nu = 0.3$, are shown in Table 3, the figures in brackets denoting the number of terms used. The convergence of the Multiple Fourier results are slow but uniform and appear to asymptote to the concentrated moment distribution results.

These examples suggest that the method is at least as good as other boundary methods such as least squares and boundary collocation, and when difficulties arise with the type of series solution adopted, slow convergence without instabilities will be indicated. The method is also readily applicable to other boundary value problems with known series solutions within star shaped regions. For example, an elastically supported plate has Bessel function series solutions each of which and its derivatives may be converted into Fourier series in θ.

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Pressures on Boat-Tailed Afterbodies in Transonic Flow with a Low-Thrust Jet

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Introduction

The pressure over the base of axisymmetric afterbodies varies significantly with Mach number in transonic streams for both cylindrical[1,2] and boat-tailed[3] afterbodies and is known to be influenced by a propulsive jet.[2] Base drag has been reduced by ejecting gas at low flow rates into the base region of a family of boat-tailed afterbodies in both

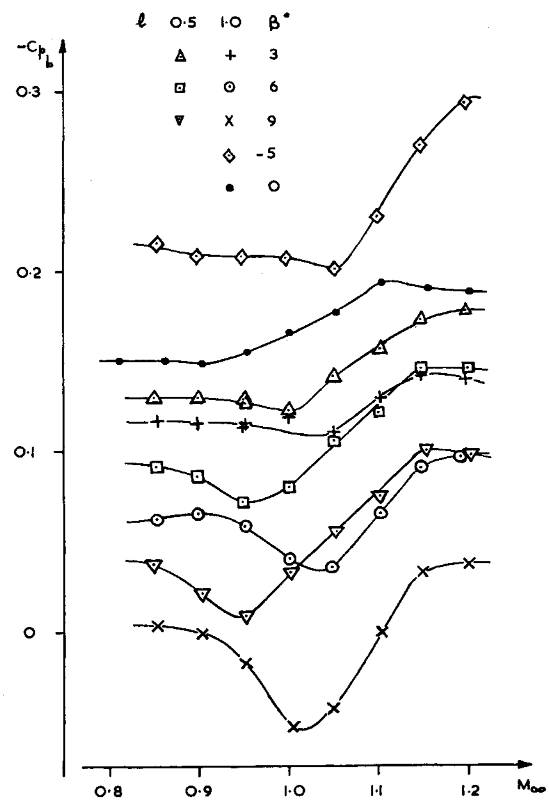

Fig. 1 Base pressure coefficient of various afterbodies.

Received November 5, 1970; revision received January 4, 1971. This work was done while the author was acting as a Vacation Consultant with the British Ministry of Defence. British Crown Copyright reserved. Published with the permission of the Controller of Her Britannic Majesty's Stationery Office.

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